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Mirror Symmetry and Landau Ginzburg Calabi-Yau Superpotentials in F-theory Compactifications

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Abstract

We study Landau Ginzburg (LG) theories mirror to 2D $N = 2$ gauged linear sigma models on toric Calabi-Yau manifolds. We derive and solve new constraint equations for Landau Ginzburg elliptic Calabi-Yau superpotentials, depending on the physical data of dual linear sigma models. In Calabi-Yau threefolds case, we consider two examples. First, we give the mirror symmetry of the canonical line bundle over the Hirzebruch surfaces $\mathbf{F_n}$. Second, we find a special geometry with the affine so(8) Lie algebra toric data extending the geometry of elliptically fibered K3. This geometry leads to a pure $N = 1$ six dimensional SO(8) gauge model from the F-theory compactification. For Calabi-Yau fourfolds, we give a new algebraic realization for ADE hypersurfaces.

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Contents

1 Introduction

One of the most beautiful properties of type II superstrings is that type IIA string propagating on a Calabi-Yau M may behave identically with type IIB string propagating on a different Calabi- Yau W. In this way, the complex (Kahler) structure moduli space of M is identical to the Kahler (complex) structure moduli space of W. The pairs of manifolds satisfying this map are known as mirror pairs, and this string duality is called mirror symmetry [1, 2, 3]. This symmetry plays also an important role in the geometric engineering of 4D $N = 2$ quantum field theories (QFT), embedded in type II superstring theories on singular Calabi-Yau threefolds, where this map can be used to obtain exact results for the type IIA superstring Coulomb branch [4, 5, 6, 7, 8].
Recently, mirror symmetry has been used in the context of 2D superconformal field theories with boundaries involving $N = 2$ sigma models (SM) and Landau -Ginzburg (LG) theories. It was shown in [9, 10, 11] how these models can be related by mirror symmetry. This leads to the map between A-type branes, wrapping special Lagrangian submanifolds, in the sigma model approach and B-type branes wrapping holomorphic cycles in the context of LG theories. This link has been a powerful tool in the study of algebraic realizations of Calabi-Yau manifolds. In particular, elliptic geometries used in the derivations of non perturbatif superstring solutions from either D-brane physics or F-theory compactifications [12, 13].

The aim of this paper is to contribute in this program by deriving new classes of constraint equations for LG elliptic Calabi-Yau superpotentials using the recent derivation of mirror symmetry in toric sigma model. In particular, we find a special elliptic Calabi-Yau threefolds extending the mirror superpotentials of the blow-up of the affine ADE local $K3$, used in the geometric engineering of $4D$ $N = 2$ superconformal field theories [5, 8]. This involves the affine so(8) Lie algebra as Mori vectors toric data leading to a new $N = 1$ SO(8) gauge model in six dimensions from F-theory compactifications point of view.
The organization of this paper is as follows. In section 2, we give an overview of Vafa's construction of F-theory. Then we study F-theory on Calabi-Yau spaces with elliptic geometric structures and the role they play in the understanding of the lower dimensional superstring models. In section 3, we first discuss aspects of $2D$ $N = 2$ linear sigma model. Second we study the interplay between this model and toric geometry which plays a crucial role for us later in this paper. We also introduce mirror symmetry, as made in [9, 10, 11], to obtain the LG mirror theory. Then we illustrate the example of elliptic $K3$ with ADE singularities, to engineer $N = 1$ gauge theories in eight dimensions with ADE gauge groups from F-theory

compactifications. In section 4, we consider two examples of the mirror symmetry for sigma model on Calabi-Yau threefolds. First, we give the mirror theory of the linear sigma model on the canonical line bundle over the Hirzebruch surfaces $\mathbf{F_n}$. This geometry recovers the leading example of $\mathbf{F_0}$ studied in the context of the mirror action of Lagrangian D-branes [11]. Second we find a special elliptic and $K3$ fibered Calabi-Yau threefolds extending the elliptic ADE mirror superpotentials to Calabi-Yau threefolds with affine so(8) Lie algebra toric data. This background space gives, from F-theory compactifications, a new pure $N=1$ SO(8) Yang-Mills theory in six dimensions. In section 5, we use the techniques developed in section 3 to derive a solution for the mirror superpotentials associated to ADE Calabi-Yau fourfolds hypersurfaces. This gives a toric realization of ADE Calabi-Yau fourfolds hypersurfaces studied in [14] in the context of derivations of $2D$ superconformal field theories from superstring compactifications. In section 6 we give our conclusion.

2 Generalities on F-theory

2.1 Review on Vafa's construction of F-theory

F-theory defines a non perturbative vacua of type IIB superstring theory in which the dilaton (ϕ) and the axion (χ) fields of the superstring are not constants. These fields are known as the complex string coupling moduli $\tau_{IIB} = \chi + ie^{-\phi}$ which is interpreted as the complex parameter of an elliptic curve leading then to non pertubative vacua of type IIB superstring theory in a twelve dimensional spacetime [12]. F-theory may be also defined by help of superstring dualities. As we will see later on, F-theory on elliptically fibered Calabi-Yau spaces may also be defined in terms of dual superstring models, but let us first review briefly some features of this theory. Type IIB is a ten dimensional theory of closed superstrings with chiral $N=2$ supersymmetry. The bosonic fields of the corresponding low energy field theory are the graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton ϕ coming from the NS-NS sector and the axion χ, the antisymmetric tensor fields $\tilde{B}_{\mu\nu}$ and the self dual four form $D_{\mu\nu\sigma\lambda}$ coming from R-R sector. As we see, there is no non abelian gauge field in the massless spectrum of type IIB superstring theory but instead contains Dp-branes solitonic objects, with $p = -1, 1, 3, 5, 7$ and 9, on which live A_μ gauge fields of open string field theory [15]. These extended objects are non perturbative solutions playing a crucial role in string dualities, and in the embedding of QFT's in superstring models by using either Hanany-Witten method [16, 17] or geometric engineering approach [4, 5, 6, 7, 8, 18]. Type IIB superstring theory has a non perturbative SL(2,Z) symmetry for which the fields $g_{\mu\nu}$ and $D_{\mu\nu\sigma\lambda}$ are invariant but the complex string coupling $\tau_{IIB} = \chi + ie^{-\phi}$ and the doublet $(B_{\mu\nu}, \tilde{B}_{\mu\nu})$ of two forms are believed to transform as [19] :

$$\tau_{IIB} \rightarrow \frac{a\tau_{IIB} + b}{c\tau_{IIB} + d}, \quad a, b, c, d \in \mathbf{Z}, \tag{2.1}$$

and

$$\begin{pmatrix} B_{\mu\nu} \\ \tilde{B}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_{\mu\nu} \\ \tilde{B}_{\mu\nu} \end{pmatrix}, \tag{2.2}$$

where the integers a, b, c and d are such that $ad - bc = 1$.

Following Vafa [12], one may interpret the complex field τ_{IIB} as the complex structure τ_{T^2} of an extra torus T^2:

$$\tau_{IIB} = \chi + ie^{-\phi} = \tau_{T^2}. \tag{2.3}$$

This extra torus T^2 combines with the ten spacetime dimensions to give a twelve dimensional theory. From this view, $10D$ type IIB superstring theory may be seen as the compactification of F-theory on the elliptic curve T^2:

$$\text{Type IIB superstring theory } \sim \frac{\text{F-theory}}{T^2}. \tag{2.4}$$

2.2 F-theory compactifications and string dualities

Here we study the F-theory compactifications and their connections to string models. To do so, we consider first a $(n + 1)$-dimensional Calabi-Yau manifold W_{n+1} which has an elliptic fibration over a n-dimensional complex base space B_n

$$y^2 = x^3 + f(z_i)x + g(z_i), \quad z_i \in B_n, \tag{2.5}$$

where z_i are the local coordinates of B_n. F-theory compactification on W_{n+1} is equivalent to type IIB superstring theory on B_n with the varying complex string coupling τ_{IIB}:

$$\chi(z_i) + ie^{-\phi(z_i)} = \tau_{T^2}(z_i). \tag{2.6}$$

The positions of the degenerate elliptic fibers on B_n are given by the solution of the following equation

$$\delta = 27g^2(z_i) + 4f^3(z_i) = 0, \tag{2.7}$$

where δ is the discriminant of the elliptic fibration. Recall that the well known example of F-theory compactification is the eight dimensional model [12, 13]. This is obtained by the compactification on elliptically fibered $K3$ surface

$$y^2 = x^3 + f(z)x + g(z), \quad z \in \mathbf{P^1} \tag{2.8}$$

In this case, the functions f and g are polynomials of degree 8 and 12 in z respectively:

$$
\begin{aligned}
f(z) &= \sum_{i=0}^{8} a_i z^i \\
g(z) &= \sum_{i=0}^{12} b_i z^i.
\end{aligned}
\tag{2.9}
$$

The complex structure τ_{T^2} is now a function of one variable z (local coordinate of $\mathbf{P^1}$) which varies over the $\mathbf{P^1}$ base of elliptically fibered $K3$. Equation (2.7) has generically 24 singular points corresponding to the neutrality condition of the discriminant δ. These singularities have a remarkable physical interpretation. To each one of the 24 points, it is associated to the location of a D7-brane of non perturbative type IIB superstring theory.

F-theory /heterotic duality

F-theory on elliptically fibered $K3$ leads to new type IIB superstring theory solutions in eight dimensions. This model is conjectured to be dual to the heterotic superstring theory on T^2,

$$
\frac{\text{F-theory}}{K3} \sim \frac{\text{heterotic superstring}}{T^2}, \tag{2.10}
$$

with the heterotic string coupling constant g_s^h is given by the size of the $\mathbf{P^1}$ base of elliptically fibered $K3$. This eight dimensional model can be further compactified to lower dimensions by fibering both sides over the same complex base B_{n-1} using the so-called adiabatic principle [20]. In this way, the above eight dimensional duality becomes [21, 22, 23]

$$
\frac{\text{F-theory}}{W_{n+1}} \sim \frac{\text{heterotic superstring}}{Z_n}, \tag{2.11}
$$

where W_{n+1} has $K3$ fibration over B_{n-1} (with $K3$=W_2). It also has an elliptic fibration, inherited from the elliptic fibration of W_2, over B_n . While the heterotic Calabi-Yau manifold Z_n is an elliptic fibration over the base B_{n-1}. For example, if we fiber eight dimensional data over an extra torus $B_1 = T^2$, then the resulting duality becomes a duality between F-theory on K3$\times T^2$ and heterotic string on T^4. The latter is known to be dual to type IIA string on K3 [24, 25, 26, 27, 28]. Thus, interesting superstring model in lower dimensions can be obtained from F-theory compactifications on elliptically fibered Calabi-Yau manifolds. This gives a pure geometric interpretation of the perturbative superstring spectrum and determines at the same time the non perturbative dynamics associated to D-brane physics in type II superstring theories or to singular bundle of $N = 1$ superstring models. In what follows we shall use the sigma model/LG mirror correspondence to develop new algebraic realization of elliptic Calabi-Yau manifolds involving both elliptic fibration and $K3$ fibration. Special attention will be given to LG elliptic Calabi-Yau 3-4 folds superpotentials.

3 Mirror symmetry in $2D$ $N = 2$ field theory

3.1 $2D$ $N = 2$ Sigma model

In this section we study the gauged linear sigma model introduced by Witten as a field theoretic description of Calabi-Yan manifolds [29]. Then we discuss the corresponding LG mirror theory studied in [9, 10, 11]. For simplicity, we consider an abelian gauge group $U(1)^r$ described by superfields V_a ($a = 1, \ldots, r$). We assume that there are k charged chiral superfields Φ_i ($i = 1, \ldots, k$) of vector charges q_i^a ($a = 1, \ldots, r$) [29, 30]. The Lagrangian of this model, in terms of superfields language, reads as

$$\mathbf{L} = \int d^2x d^4\theta \sum_{i=1}^{k} \bar{\Phi}_i e^{2q_i^a V_a} \Phi_i - \sum_a \rho_a \int d^2x d^4\theta V_a + (\int d^2x d^2\theta W(\Phi) + hc). \tag{3.1}$$

Integrating with respect to θ, we find the superpotential energy for the dynamical scalar fields ϕ_i

$$U(\phi) = \sum_{i=1}^{k} |\frac{\partial W}{\partial \phi_i}|^2 + \sum_{a=1}^{r} \frac{1}{2e_a^2} D_a^2, \tag{3.2}$$

where W is the superpotential and e_a's are the gauge coupling parameters. D^a are known as D-terms:

$$D_a^2 = \sum_{a=1}^{r} e_i^2 (\sum_{i=1}^{k} q_i^a |\phi_i|^2 - \rho_a)^2, \tag{3.3}$$

where the ρ_a parameters are Fayet-Illiopoulos (FI) terms with the θ angles give complexified Kahler parameters:

$$t_a = \rho_a + i\theta_a, \quad a = 1, \ldots, r. \tag{3.4}$$

We next suppose that there is no superpotential for the charged matters

$$W(\phi) = 0.$$

Thus, the minimum of the potential energy comes only from the D-terms. Vanishing of these terms give us

$$\sum_{i=1}^{k} q_i^a |\phi_i|^2 = \rho_a. \tag{3.5}$$

Dividing the space of solutions (which will be called M) of (3.5) by the complexified gauge group $U(1)^r$

$$\phi_i \rightarrow e^{iq_i^a \gamma_a} \phi_i, \tag{3.6}$$

we find the following complex space

$$\frac{C^k}{C^{*r}}, \tag{3.7}$$

where C^k corresponds to complex coordinates z_i and the C^* actions are given by

$$z_i \longrightarrow \lambda^{q_i^a} z_i, \quad i = 1, 2, \ldots, k; \;\; a = 1, 2, \ldots, r. \tag{3.8}$$

For example, if we have a $U(1)$ gauge theory with two chiral fields with charges 1, the classical moduli space is $\mathbf{P^1}$.

The space of solutions (M) we have been describing has a nice geometrical interpretation in terms of toric geometry [1]. This has been a beautiful interplay between $2D$ $N = 2$ sigma models and toric geometry [29, 5]. Indeed interpreting the previous (ϕ_i) matter fields as the z_i coordinates of (3.7) and the q_i^a quantum charges, under the $U(1)^r$ symmetry, should be interpreted as the Mori vectors of toric geometry language. In this way, the vacuum space may have a toric diagram Δ which consists of k vertices $\{v_i\}$ in the standard lattice $\mathbf{Z^n}$, where $n = k - r$ is the complex dimension of the space of solutions (we are assuming that there is no toric fibration structure). Every vertex v_i corresponds to a matter field (ϕ_i) in our $2D$ $N = 2$ sigma model. Since the complex dimension of vacuum space is n, there are r relations between the k vertices which read as

$$\sum_{i=1}^{k} q_i^a v_i = 0, \quad a = 1, \ldots, r. \tag{3.9}$$

In this representation, it is worthwhile to mention the four following:

1- If the q_i^a's are all positive definite, or negative definite, the space of solutions is compact. However, if there is a mixture of positive and negative q_i^a's, the toric target space is non compact .

2- For q_i^a's obeying the neutrality condition

$$\sum_{i=1}^{k} q_i^a = 0, \quad a = 1, \ldots, r, \tag{3.10}$$

the toric target space is a non compact Calabi-Yau manifold and the field theory flows in the infrared to a non trivial superconformal model [29, 5, 37, 38]. This type of manifolds plays a crucial role in the study of non perturbative superstring theory compactifications, in particular, in the geometric engineering of QFT's.

3- If all ρ_a's are zero, then the toric manifold is singular.

4- For all ρ_a's $\neq 0$, we have a smooth toric manifold. In this case the (FI) parameters, which are given by the Hodge number $h^{1,1}(M)$ or equivalently by the number of $U(1)$ factors, are interpreted as blow up parameters of the singularity.

[1] For more details on toric geometry, see [31, 32, 33, 34, 35, 36].

3.2 LG mirror theory

Having introduced the linear sigma model construction, we will now discuss the corresponding mirror theory. There are different ways one follow to obtain the mirror theory. The latter is a LG model with Calabi-Yau superpotentials, depending on the number of chiral multiples and gauge fields of dual theories. A tricky way, to write down the equation of LG mirror superpotential (dual to the previous Sigma model (3.5)), is to introduce in the game k dual chiral fields Y_i to each field in the sigma model such that [9, 10, 11]

$$\mathrm{Re}\, Y_i = |\phi_i|^2, \quad i = 1, \ldots, k. \tag{3.11}$$

For convenience, we define new variables x_i

$$x_i = e^{-Y_i}. \tag{3.12}$$

The defining equation of the LG mirror superpotential takes the form

$$\sum_{i=1}^{k} x_i = 0, \tag{3.13}$$

where the fields x_i must satisfy

$$\prod_{i=1}^{k} x_i^{q_i^a} = e^{-t_a}, a = 1, \ldots, r. \tag{3.14}$$

Recall that the t_a are the complexified Kahler parameters of sigma models which now define the complex structure of LG mirror geometry. The solution of these equations often described by $(n-2)$-dimensional hypersurfaces. This is not a problem since one can restore the correct dimension by introducing two auxiliary fields u and v and equation (3.13) becomes

$$W(x_i) = \sum_{i=1}^{k} x_i = uv \tag{3.15}$$

Note that the quadratic term uv does not affect the complex structure of the mirror superpotential.

3.3 Elliptic ADE mirror superpotentials

As our first example we consider the LG $K3$ superpotentials with deformed elliptic ADE singularities. We start by constructing these geometries, with ADE singularities, as gauged $N = 2$ two dimensional linear sigma model. In general, these are described by a $U(1)^{r+1}$ gauge group with $(r+5)$ chiral multiples ϕ_i with q_i^a vector charges specified latter on. The elliptic ADE spaces of classical vacua, in the absence of the sigma model superpotential, are given by

$$U = \sum_{a=0}^{r} e_a^2 (\sum_{i=1}^{r+5} q_i^a |\phi_i|^2 - \rho_a)^2, \quad a = 0, 1 \ldots, r, \tag{3.16}$$

where r is the rank of ADE algebras and the q_i^a's are the quantum charges of ϕ_i under the $U(1)^{r+1}$ gauge symmetry, up to details are proportional to the Cartan matrices K_{ai} of the ADE Lie algebras in question. These vectors charges satisfy the condition $\sum_{i=1}^{r+5} q_i^a = 0$ under which the gauge model flow in the infrared to 2D $N = 2$ superconformal field theory. The ADE spaces of classical vacua may be described by a toric diagram Δ spanned by $(r+5)$ vertices

$$v_i = (n_i, m_i, s_i) \tag{3.17}$$

of the standard lattice $\mathbf{Z^3}$, where the first entry n_i takes either zero or the Dynkin weighted of the adjoint representation of the corresponding Lie algebra. These $(r+5)$ vertices fulfill the following $(r+1)$ relations:

$$\sum_{i=1}^{r+5} q_i^a v_i = 0, \quad a = 0, 1..., r, \tag{3.18}$$

with the Calabi-Yau condition

$$\sum_{i=1}^{r+5} q_i^a = 0. \tag{3.19}$$

Having introduced the toric data of sigma model construction of local elliptic ADE $K3$ surface, we will now apply the mirror symmetry to get the corresponding ADE superpotentials for LG theory. To write down the algebraic equations of the mirror geometry, we will use the toric data of sigma model construction. Indeed, we associate to each vertex v_i of the toric diagram Δ a monomial $x_1^{n_i} x_2^{m_i} x_3^{s_i}$, where x_1, x_2 and x_3 are LG gauge invariant fields [8, 36]. The superpotentials of the mirror theory associated with the vertices (3.17) are described by complex $2D$ Calabi- Yau surfaces $W_2(ADE)$:

$$W_2(ADE) = \sum_{i=1}^{r+5} a_i {x_1}^{n_i} {x_2}^{m_i} {x_3}^{s_i} = 0, \tag{3.20}$$

where a_i are the complex parameters defining the complex structure of the mirror superpotentials. Note that only a subset of a_i are physical. Recall that the fields x_i may be viewed as gauge invariant under the C^* action of weighted projective spaces $\mathbf{WP^3}$, in which the elliptic $K3$ is embedded [5, 8, 36], allowing us to give a homogeneous description of elliptic ADE series for LG mirror superpotentials. This homogeneous description takes the form

$$W_2(ADE) = P_0(y, x, z) + \sum_i w^i P_i(y, x, z) = 0, \tag{3.21}$$

where (y, x, z, w) are the homogeneous coordinates of $\mathbf{WP^3}(3, 2, 1, \eta)$ and where η is an integer depending on the type of Lie algebras [8, 36]. P_0 describes an elliptic curve $\mathbf{E}$ in $\mathbf{WP^2}$. In particular, a sextic in $\mathbf{WP^2}(3, 2, 1)$

$$P_0 = y^2 + x^3 + z^6 + \mu xyz = 0, \tag{3.22}$$

with μ is a complex structure moduli. The number of monomials in P_i is equal to the number of Dynkin labels equal to i. Equation (3.21) gives the ADE mirror classification of local elliptically fibered $K3$ and plays a crucial role in geometric engineering of $N = 1$ gauge theories in eight dimensions. Indeed, at singular limit of F-theory on W_2, when the 2-cycles of $K3$ shrink to zero area, we get enhanced $N = 1$ ADE gauge symmetries in eight dimensions [21, 22, 23, 39, 36]. Recall that the resolution of ADE singular $K3$ occurring in F-theory compactifications consists of affine Dynkin diagrams of a chain of 2-cycles with specific intersection numbers in agreement with the corresponding affine Dynkin index. For illustration, let us present an example concerning the elliptic affine A_{n-1} space (for n even). The toric data of this geometry has four vertices:

$$v_0 = (0, 0, 0), v_1 = (0, 2, 3), v_2 = (0, -1, 0), v_3 = (0, 0, -1), \tag{3.23}$$

describing the elliptic fiber $\mathbf{E}$ and n vertices,

$$v_1 = (1, 2, 3), v_{2i} = (1, 2 - i, 3 - i), v_{2i+1} = (1, 2 - i, 2 - i),\ i > 0, \tag{3.24}$$

introduced by the blow ups. The superpotential of the mirror LG model is

$$W_2(A_{n-1}) = (y^2 + x^3 + z^6 + \mu xyz) + wP_1(x, y, z) = 0 \tag{3.25}$$

where

$$P_1(x, y, z) = a_n z^n + a_{n-2} z^{n-2} x + a_{n-3} z^{n-3} y + \ldots + a_0 x^{\frac{n}{2}}. \tag{3.26}$$

For completeness we give the toric vertices for others elliptic geometries

-$\hat{D}_n$ *Geometry*

$$\begin{array}{l}
v_0 = (1, 2, 3), v_1 = (1, 1, 1) \\
v_{i+1} = (2, 3 - i, 4 - i), \quad i = 1, \ldots, n - 3 \\
v_{n-1} = (1, \frac{1}{2}(6 - \epsilon - n), \frac{1}{2}(6 - \epsilon - n)),\ v_n = (1, \frac{1}{2}(4 + \epsilon - n), \frac{1}{2}(6 + \epsilon - n)) \\
\tilde{v}_n = (0, 0, 0), \tilde{v}_{n+1} = (0, -1, 0), \tilde{v}_{n+2} = (0, 0, -1), \tilde{v}_{n+3} = (0, 2, 3) \\
q_i^0 = (-2, 0, 1, 0^{n-2}, 0, 0, 0, 1),\ q_i^1 = (0, 1, -2, 1, 0^{n-4}, 0, 0, 0, 0,) \\
q_i^2 = (1, 1, -2, 1, 0^{n-3}, -1, 0, 0, 0), \ldots \\
q_i^{n-1}(\epsilon = 0) = (0^{n-2}, 1, -2, 0, 0, 0, 1, 0),\ q_i^{n-1}(\epsilon = 1) = (0^{n-2}, 1, -2, 0, -2, 1, 2, 0), \\
q_i^n(\epsilon = 0) = (0^{n-2}, 1, 0, -2, -2, 2, 1, 0),\ q_i^n(\epsilon = 1) = (0^{n-2}, 1, 0, -2, 0, 1, 0, 0), i = 0, \ldots, n + 4
\end{array} \tag{3.27}$$

where $\epsilon = 0$ (1) for n even (old).

$-\hat{E}_6$ (curve **E** in P^2) :

$$\begin{aligned}
& v_0 = (1,-1,1),\ v_1 = (1,-1,-1), v_2 = (2,-1,0),\ _3 = (2,-1,-1) \\
& _4 = (3,-1,-1),\ v_5 = (2,0,-1),\ v_6 = (1,1,-1) \\
& \tilde{v}_7 = (0,0,0), \tilde{v}_8 = (0,2,-1),\ \tilde{v}_9 = (0,-1,2),\ \tilde{v}_{10} = (0,-1,-1) \\
& q_i^0 = (-2,0,1,0,0,0,0,0,0,1,0),\ q_i^1 = (0,-2,0,1,0,0,0,0,0,0,1) \\
& q_i^2 = (1,0,-2,0,1,0,0,0,0,0,0) \\
& q_i^3 = (0,1,0,-2,1,0,0,0,0,0,0) \\
& q_i^4 = (0,0,1,1,-2,1,0,-1,0,0,0) \\
& q_i^5 = (0,0,0,0,1,-2,1,0,0,0,0) \\
& q_i^6 = (0,0,0,0,1,-2,1,0,0).
\end{aligned} \tag{3.28}$$

$-\hat{E}_7$ (curve **E** in $WP^2(1,1,2)$) :

$$\begin{aligned}
& v_0 = (1,-2,1),\ v_1 = (2,-2,-1),\ v_2 = (2,-1,0),\ v_3 = (3,-2,-1) \\
& v_4 = (4,-2,-1),\ v_5 = (3,-1,-1),\ v_6 = (2,0,-1),\ v_7 = (1,1,-1) \\
& \tilde{v}_8 = (0,0,0),\ \tilde{v}_9 = (0,2,-1),\ \tilde{v}_{10} = (0,0,1),\ \tilde{v}_{11} = (0,-2,-1) \\
& q_i^0 = (-2,1,0,0,0,0,0,0,0,0,0,1),\ q_i^1 = (1,-2,0,1,0,0,0,0,0,0,0,0) \\
& q_i^2 = (0,0,-2,0,1,0,0,0,0,0,1,0) \\
& q_i^3 = (0,1,0,-2,1,0,0,0,0,0,0,0) \\
& q_i^4 = (0,0,1,1,-2,1,0,0,-1,0,0,0) \\
& q_i^5 = (0,0,0,0,1,-2,1,0,0,0,0,0) \\
& q_i^6 = (0,0,0,0,0,1,-2,1,0,0,0,0) \\
& q_i^7 = (0,0,0,0,0,0,1,-2,0,1,-2,0,1,0,0).
\end{aligned} \tag{3.29}$$

$-\hat{E}_8$ (curve **E** in $WP^2(1,2,3)$):

$$\begin{aligned}
& E_8 : \\
& v_0 = (1,2,3),\ v_1 = (2,2,3),\ v_2 = (3,1,1),\ v_3 = (3,2,3) \\
& v_4 = (4,2,3),\ v_5 = (5,2,3),\ v_6 = (6,2,3),\ v_7 = (4,1,2),\ v_8 = (2,0,1) \\
& \tilde{v}_9 = (0,0,0), \tilde{v}_{10} = (0,-1,0), \tilde{v}_{11} = (0,0,-1), \tilde{v}_{12} = (0,2,3) \\
& q_i^0 = (-2,1,0,0,0,0,0,0,0,0,0,0,1) q_i^1 = (1,-2,0,1,0,0,0,0,0,0,0,0,0) \\
& q_i^2 = (0,0,-2,0,0,0,1,0,0,0,0,1,0) \\
& q_i^3 = (0,1,0,-2,1,0,0,0,0,0,0,0,0) \\
& q_i^4 = (0,0,0,1,-2,1,0,0,0,0,0,0,0) \\
& q_i^5 = (0,0,0,0,1,-2,1,0,0,0,0,0,0) \\
& q_i^6 = (0,0,1,0,0,1,-2,1,0,-1,0,0,0) \\
& q_i^7 = (0,0,0,0,0,0,1,-2,1,0,0,0,0) \\
& q_i^8 = (0^7,-2,0,1,0,0).
\end{aligned} \tag{3.30}$$

-BCFG models

We conclude this section by noting that the above analysis is also valid for non simply laced

BCFG LG mirror geometries of $K3$. This is based on toric realizations of the standard techniques of the folding of the Dynkin nodes of ADE graphs which are permuted by outer-automorphism groups [8, 36]. Indeed, starting form the toric data of the ADE simply laced geometries considered above one gets the constraint equations of the folding of toric vertices of non simply laced geometries using the well known results:

$$\begin{array}{ll} D_{n+1}/Z_2 & \to B_n \\ A_{2n-1}/Z_2 & \to C_n \\ E_6/Z_2 & \to F_4 \\ D_4/Z_2 & \to G_2. \end{array} \tag{3.31}$$

4 LG Calabi-Yau threefolds superpotentials

In this section we will describe the LG theory mirror to sigma model on toric Calabi-Yau threefolds and the role they play in the description of F-theory vacua in six dimensions. In particular, supersymmetric QFT's limit of low effective models of F-theory on singular Calabi-Yau threefolds. We do not attempt to give a classification, but instead we will consider two examples. We first study the LG theory mirror to linear sigma model on the canonical line bundle over the Hirzebruch surfaces $\mathbf{F_n}$. This geometry recovers the leading example of $\mathbf{F_0}$ studied in the context of the mirror action of Lagrangian D branes [11]. We will see that the mirror geometry has also an elliptic fibration structure

$$f(x_1, x_2) = uv. \tag{4.1}$$

where $f(x_1, x_2) = 0$ describes a Riemann surface, x_1, x_2 are C^* coordinates and u, v are C coordinates. Second we will consider a special mirror geometry extending the $K3$ mirror superpotantials with ADE singularities studied in section 3.

4.1 Elliptic fibration models

Let us start with the first example describing the sigma model on the canonical line bundle over the Hirzebruch surfaces $\mathbf{F_n}$, $(\mathbf{n \geq 0})$. Recall by the way that the $\mathbf{F_n}$ geometries are defined by a non-trivial fibration of a $\mathbf{P^1}$ fiber on a $\mathbf{P^1}$ base. These geometries are realized as the vacuum manifold of the $U(1) \times U(1)$ gauge theory with four chiral fields with charges

$$\begin{array}{rcl} q_i^{(1)} & = & (1, 1, 0, -n) \\ q_i^{(2)} & = & (0, 0, 1, 1). \end{array} \tag{4.2}$$

These surfaces have a nice realization in terms of toric geometry techniques [31]. This represented by four vertices in $\mathbf{Z^2}$ as follows

$$
\begin{aligned}
v_1 &= (1,0) \\
v_2 &= (-1,n) \\
v_3 &= (0,1) \\
v_4 &= (0,-1).
\end{aligned}
\tag{4.3}
$$

These vertices satisfy the following linear toric relations

$$
\begin{aligned}
v_1 + v_2 + n v_4 = 0 \\
v_3 + v_4 = 0.
\end{aligned}
\tag{4.4}
$$

Note that the $\mathbf{F_n}$ surfaces are not Ricci-flat. However they can be viewed as a part of a local geometry of a Calabi-Yau manifold, where there are extra dimensions. In particular, if we embed these surfaces in a Calabi-Yau 3-folds there is a normal direction corresponding to line bundle on $\mathbf{F_n}$. The Calabi-Yau condition requires that the normal bundle must be a canonical line bundle. Thus the canonical line bundle over $\mathbf{F_n}$ are local Calabi-Yau threefolds [2]. These geometries are used in superstring theory compactifications, in particular, in the geometric engineering of $4D$ $N=2$ supersymmetric gauge theories, where these background spaces allow us to rederive the Seiberg-Witten models [40, 41]. Roughly speaking, the canonical line bundle of $\mathbf{F_n}$ surfaces is described by a $U(1)\times U(1)$ linear sigma model with five matter fields ϕ_i with two vector charges

$$
\begin{aligned}
q_i^{(1)} &= (1,1,0,-n,n-2) \\
q_i^{(2)} &= (0,0,1,1,-2).
\end{aligned}
\tag{4.5}
$$

The D-flatness conditions of this model read as

$$
\begin{aligned}
|\phi_1|^2 + |\phi_2|^2 - n|\phi_4|^2 + (n-2)|\phi_5|^2 &= \rho_1 \\
|\phi_3|^2 + |\phi_4|^2 - 2|\phi_5|^2 &= \rho_2.
\end{aligned}
\tag{4.6}
$$

This classical vacuum has a geometrical realization in terms of the following toric data

$$
\sum_{i=1}^{5} q_i^a v_i = 0, \quad \sum_{i=1}^{5} q_i^a = 0,
\tag{4.7}
$$

[2]Recall that for the leading example corresponding to $\mathbf{F_0} = \mathbf{P^1}\times\mathbf{P^1}$ (trivial fibration), the canonical line bundle of $\mathbf{F_0}$ looks like as the A_1 ALE space (local $K3$ surface) fibered over a $\mathbf{P^1}$ base space [5].

where the vertices v_i, which are elements of the standard lattice $\mathbf{Z^3}$, are given by

$$v_1 = (1,0,1),\ v_2 = (-1,n,1),\ v_3 = (0,1,1),\ v_4 = (0,-1,1),\ v_5 = (0,0,1), \tag{4.8}$$

and $\sum_{i=1}^{5} q_i^a = 0$ is the Calabi-Yau condition to ensure the cancellation of the first Chern class $c_1 = 0$. Using equations (3.13-14), the LG mirror superpotential is obtained by solving the following constraint equations

$$\begin{aligned} W_3(x_i) &= x_1 + x_2 + x_3 + x_4 + x_5, && (4.9)\\ x_1x_2 &= e^{-t_1}x_4^n x_5^{2-n} && (4.10)\\ x_3x_4 &= e^{-t_2}x_5^2. && (4.11)\end{aligned}$$

After a direct computation in the patch $x_5 = 1$, we get

$$f_n(x_1,x_4) = 1 + x_1 + \frac{e^{-t_1}x_4^n}{x_1} + x_4 + \frac{e^{-t_2}}{x_4} = 0. \tag{4.12}$$

This LG mirror geometry has naively 1-dimensional Riemman surface. This not a problem since the LG mirror superpotential encodes all the informations of sigma model physical Kahler parameters; and one can restore the correct dimension, which is 3, by introducing the irrelevant quadratic term uv in equation (4.9). Thus, the defining equation for the LG mirror superpotential becomes

$$\begin{aligned} W_3(x_i,u,v) &= f_n(x_1,x_4) - uv = 0\\ &= 1 + x_1 + \frac{e^{-t_1}x_4^n}{x_1} + x_4 + \frac{e^{-t_2}}{x_4} - uv = 0, && (4.13)\end{aligned}$$

which now describes a non compact toric Calabi-Yau 3-folds, moreover, permits us to go beyond the $\mathbf{F_0}$ case used in [11]. Equation (4.12) implies that this geometry has an elliptic fibration model over $\mathbf{C^2}$ with coordinates u et v whose the fiber is a Riemann surface

$$f_n(x_1,x_4) = 0; \tag{4.14}$$

where for $n = 0$ we have an elliptic curve in the $\mathbf{P^2}$ projective space. To see this, we shall proceed in two steps as follows. First, we consider the LG fields x_1 and x_4 as two invariant gauge fields under C^* action of the $\mathbf{P^2}$ in which the fiber is embedded:

$$\begin{aligned} x_1 &= \frac{x}{z}\\ x_4 &= \frac{y}{z}, && (4.15)\end{aligned}$$

where x, y and z are the homogeneous variables of the $\mathbf{P}^2$

$$(x, y, z) \to (\lambda x, \lambda y, \lambda z).$$

Second, putting the equation (4.15) in (4.13) for $n = 0$

$$f_0(x_1, x_2) = 1 + x_1 + \frac{e^{-t_1}}{x_1} + x_4 + \frac{e^{-t_2}}{x_4}; \tag{4.16}$$

and multiplying by xyz, we get the homogeneous description of the elliptic fiber

$$f_0(x, y, z) = x^2 y + xy^2 + e^{-t_1} yz^2 + e^{-t_2} xz^2 + xyz. \tag{4.17}$$

This equation is a cubic polynomial in $\mathbf{P^2}$ whose the general form is given by

$$\sum_{i+j+k=3} a_{ijk} x^i y^j z^k = 0, \tag{4.18}$$

which can be written in the following Weierstrass form

$$y^2 z = x^3 + axz^2 + bz^3. \tag{4.19}$$

As we have seen, this form plays an important role in the discussion of elliptic Calabi-Yau manifolds involved in F-theroy to derive non perturbative vacua of type IIB superstring.

4.2 K3 fibration in F-theory compactifications

4.2.1 LG $K3$ fibration Calabi-Yau superpotential

Our second example of Calabi-Yau threefolds is quite similar to the first one, and our treatment of it will parallel to the above discussion. This example of model will be given by the LG mirror superpotential with a local toric Calabi-Yau 3-fold configuration, which is both elliptic and $K3$ fibration. Roughly speaking, the dual field content of $2D$ $N = 2$ linear sigma model is a $U(1)^5$ supersymmetric gauge theory with ten (ϕ_i) matter fields with vectors charges q_i^a ($a = 0, \ldots, 4$). The latters are the quantum charges of the (ϕ_i)'s under the corresponding $U(1)^5$'s:

$$\begin{aligned} q_i^0 &= (-2, 0, 1, 0, 0, 1, 0, 0, 0, 0) \\ q_i^1 &= (0, -2, 1, 0, 0, 0, 0, 1, 0, 0) \\ q_i^2 &= (1, 1, -2, 1, 1, 0, 0, 0, 0, -2) \\ q_i^3 &= (0, 0, 1, -2, 0, 0, 0, 1, 0, 0) \\ q_i^4 &= (0, 0, 1, 0, -2, 0, 0, 0, 0, 1), \end{aligned} \tag{4.20}$$

which are, up to some details, the opposite of the affine $so(8)$ Cartan matrix $K_{ai}(so(8))$:

$$q_i^a = -K_{ai}(so(8)), \quad i = 1, \ldots, 5; \quad a = 0, \ldots, 4. \tag{4.21}$$

The space of classical vacua of this model is given by the D-flatness equations namely

$$\sum_{i=1}^{10} q_i^a |\phi_i|^2 = \rho_a, \quad a = 0, \ldots, 4, \tag{4.22}$$

This space of solutions has also a geometrical realization described by the following toric data

$$\sum_{i=1}^{10} q_i^a v_i = 0, \quad a = 0, \ldots, 4 \tag{4.23}$$

where

$$\begin{aligned} & v_1 = (1,1,-1,-1), v_2 = (1,-1,-1,1), v_3 = (2,-1,-1,-1), \\ & v_4 = (1,-1,-1,-1), v_5 = (1,-1,1,-1), v_6 = (0,3,-1,-1), \\ & v_7 = (0,-1,-1,3), v_8 = (0,-1,-1,-1), v_9 = (0,-1,3,-1), \\ & \qquad\qquad\qquad\qquad\qquad\qquad v_{10} = (0,0,0,0). \end{aligned} \tag{4.24}$$

Using equations (3.13-14) and recalling the variables, the mirror theory has superpotential

$$W_3(x_i) = \sum_{i=1}^{10} x_i = 0, \tag{4.25}$$

where the x_i's satisfy the following constraint equations,

$$\begin{aligned} x_3^2 x_{10}^2 &= x_1 x_2 x_4 x_5 \\ x_1^2 &= x_3 x_6 \\ x_2^2 &= x_3 x_7 \\ x_4^2 &= x_3 x_8 \\ x_5^2 &= x_3 x_9. \end{aligned} \tag{4.26}$$

These constraints can be solved by the monomials

$$\begin{aligned} x_1 = w z_1^2, \quad x_2 = w z_2^2, \quad x_3 = w^2, \quad x_4 = w z_3^2, \quad x_5 = w z_4^2. \\ x_6 = z_1^4, \quad x_7 = z_2^4, \quad x_8 = z_3^4, \quad x_9 = z_4^4, \quad x_{10} = z_1 z_2 z_3 z_4. \end{aligned} \tag{4.27}$$

Thus, the LG mirror superpotential is

$$W_3 = z_1^4 + z_2^4 + z_3^4 + z_4^4 + \psi z_1 z_2 z_3 z_4 + a_0 w^2 + w(a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 + a_4 z_4^2) = 0, \tag{4.28}$$

where ψ and a_i, which given in terms of t_i, are complex parameters defining the complex structure.
Equation (4.28) is invariant under the C^* action

$$(z_1, z_2, z_3, z_4, w) \rightarrow (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, \lambda^2 w), \tag{4.29}$$

and describes a 3-dimensional hypersurface in $\mathbf{WP^4}(1,1,1,1,2)$ with $c_1 \neq 0$. One easily restore the Calabi-Yau condition by considering wW_3 as the Calabi-Yau hypersurface which defines a singular 3-dimensional toric manifold. This geometry is not only elliptic but also K3 fibration. To see this, consider first the w independent terms namely

$$P_\psi = z_1^4 + z_2^4 + z_3^4 + z_4^4 + \psi z_1 z_2 z_3 z_4. \tag{4.30}$$

This defines a quartic hypersurface in $\mathbf{P^3}$ describing a $K3$ surface with a complex structure ψ. Second, we take the large complex structure limit ($\psi \rightarrow \infty$). In this appropriate limit, the equation (4.30) becomes approximately

$$P_\infty = z_1 z_2 z_3 z_4 = 0. \tag{4.31}$$

According to [33], this means that the quartic $K3$ is now a T^2 fibration over the boundary faces of the toric diagram Δ of the $\mathbf{P}^3$ projective space:

$$K3 = T^2(R_1, R_2) \times B_2, \tag{4.32}$$

where (R_1, R_2) are the two radii of the torus T^2 and $B_2 = \partial\Delta$ consists of the union of four triangles of three dimensional standard simplex. Note that this torus can degenerate over the fixed faces of the $\mathbf{P^3}$ toric action. One distinguishes two cases:
1- The torus degenerates to a circle at each edge, which means that one 1-cycle shrinks to zero size,

$$R_i = 0, \quad R_{j\neq i} \neq 0, \quad i, j = 1, 2. \tag{4.33}$$

This is the same situation appearing in the large complex structure limit of elliptic curves involved in the study of non perturbative vacua of type IIB string from F-theory compactifications on elliptic fibration manifolds.
2- The torus T^2 completely degenerates over the endpoints of each triangle, where the two 1-cycles of T^2 shrink to zero size:

$$R_i = 0, \quad i = 1, 2. \tag{4.34}$$

In these singular limits, one may take the complex structure ψ of $K3$ as

$$\psi \sim \frac{V(B_2)}{R_1 R_2}, \tag{4.35}$$

where $V(B_2)$ denotes the volume of the base space B_2. In this way, W_3 geometry may be now regarded as fibering elliptic $K3$ surface (4.30), in which the fiber has vanishing first Chern class (i.e) $c_1 = 0$, over a base space parameterized by w. Our W_3 Calabi-Yau geometry has the following nice features:

(1) It extends the geometry of the W_2, studied in section 3, to

$$\sum_i w^i P_i(z_1, z_2, z_3, z_4) = 0, \tag{4.36}$$

for an elliptic Calabi-Yau 3-folds. In other words, instead of having a curve in two dimensional projective spaces(as in the elliptic $K3$ surface (3.22)), we now have a surface

$$P_0(z_1, z_2, z_3, z_4) = 0$$

in the three dimensional projective space $\mathbf{P^3}$; where the w exponents are exactly the Dynkin index of affine $so(8)$ Lie algebra.

(2) W_3 gives a new realization of $so(8)$ Lie algebra in terms of Calabi -Yau 3-folds. This toric realisation is closed related to the standard tetravalent geometry [36], which may be viewed as higher geometry of trivalent and bivalent geometries used in the context of geometric engineering of QFT in four dimensions [5]. The latter is described by the following monomials

$$1, z_1^2, z_2^2, z_3^2, z_4^2, z_1 z_2 z_3 z_4, \tag{4.37}$$

where this geometry may be used to extend the $T_{p,q,r}$ singularity to $T_{p,q,r,t}$ by considering four intersecting SU chains.

(3) The complex structure determined by the complex parameters ψ and a_i maight be used to define a moduli space of SO(8) bundle on quartic $K3$.

4.2.2 $D = 6$ $N = 1$ SO(8) gauge theory

Having introduced the geometric background space W_3, we will now discuss the corresponding gauge theory if one consider the F-theory compactification. As well known that F-theory on $K3$ fibration Calabi-Yau manifolds are intimately related to $N = 1$ string models. In particular, six dimensional compactifications of F-theory on Calabi-Yau threefolds, where these geometries encode the informations about physical data of $N = 1$ string theories including the enhanced gauge symmetries, the perturbative matter fields and the non perturbative dynamics corresponding to small instanton singularities [42]. Roughly speaking, mimicking the analysis in [22, 23], F-theory on singular W_3 (4.27) leads to a pure $N = 1$ Yang-Mills theory in six dimensions. The corresponding gauge group, associated to this W_3 geometry, is given by the SO(8) gauge group determined by the intersection matrix of the blown up toric divisors (4.20). Moreover since the perturbative gauge symmetries in heterotic superstring models stems only from the classification of singularities of $W_2(ADE)$ fiber space, studied in section 2, this SO(8) gauge model may be related to non perturbative dynamics.

5 On ADE Calabi-Yau fourfolds superpotentials

In this section we want to extend the previous analysis to higher dimensional elliptic Calabi-Yau geometries. In particular, we will consider $(n+2)$-dimensional elliptic Calabi-Yau's, where they will be realized as n-dimensional elliptic Calabi-Yau manifolds over 2 complex dimensional base space. They may be viewed as few extensions of non compact Calabi-Yau 3-folds (4.1). These geometries may be expressed in the following form

$$f(x_1, \dots, x_{n+1}) = uv. \tag{5.1}$$

In other words, instead of having a Reimann surface as in the case of Calabi-Yau 3-folds, we now have a n-dimensional Calabi-Yau fibers

$$f(x_1, \dots, x_{n+1}) = 0. \tag{5.2}$$

These extended geometries may play a crucial role in the understanding of the lower dimensional non perturbative superstring theories.

From the F-theory compactification point of view, we will restrict to a particular case corresponding to elliptic Calabi-Yau 4-folds

$$f(x_1, x_2, x_3) = uv. \tag{5.3}$$

Before discussing the $2D$ $N = 2$ sigma model construction of these manifolds, it useful to review some basic facts about the different constructions of the Calabi-Yau 4-folds. The latters can have realizations of many types:

1- The orbifold $\frac{\mathbf{C^4}}{\mathbf{Z_4}}$:

$$z_j \to i z_j, \quad j = 1, \dots, 4. \tag{5.4}$$

2- The hyper-Kahler quotient in terms of two dimensional field theory with eight supercharges in presence of charged hypermultiples.
3 -The ADE hypersurfaces in $\mathbf{C^5}$ considered in [14] in the context of derivations of two dimensional superconformal field theories from singular limits of type IIA superstring compactifications.
Here we are interested in elliptic ADE 4-folds hypersurfaces having elliptic $K3$ toric fibration, with ADE singularities, over 2-dimensional base spaces. A priori there are different ways one may follow to give the corresponding $2D$ $N=2$ linear sigma model construction. A naive way to do this is to consider these geometries as a moduli space of two orthogonal models described by $2D$ $N=2$ supersymmetric field theories. In this method, it is possible to see the elliptic ADE $N=2$ linear sigma model, studied in section 3, as a fiber and the other model whose the target space is a two complex dimensional space, as a base. However this way of doing may bring extra parameters in the moduli space of ADE Calabi-Yau fourfolds hypersurfaces. A tricky method to overcome this problem is to use the previous elliptic ADE $(K3)$ $N=2$ linear sigma model with extra chiral fields, corresponding to the two complex dimensional base space of Calabi-Yau fourfolds. Roughly speaking, we consider the previous $U(1)^{r+1}$ linear sigma model but with $(r+8)$ chiral fields ϕ_j $(j=1,\dots,r+8)$ with matrix charge Q_j^a. The latters are given by

$$Q_j^a = (q_i^a, q_{r+6}^a, q_{r+7}^a, q_{r+8}^a), \quad i=1,\dots,r+5, \;\; a=1,\dots,r+1, \tag{5.5}$$

where q_i^a are exactly the matrix charge of $(r+5)$ chiral fields ϕ_i of $U(1)^{r+1}$ linear sigma model construction of ADE elliptic $K3$ and $(q_{r+6}^a, q_{r+7}^a, q_{r+8}^a)$ are the quantum charges of the extra fields, under $U(1)^{r+1}$ symmetry, will be specified latter on. The condition under which the gauge system flow in the infrared to $2D$ $N=2$ superconformal field theory is

$$\sum_{j=1}^{r+8} Q_j^a = 0, \quad a=0,1,\dots,r. \tag{5.6}$$

However the Calabi-Yau condition $\sum_{i=1}^{r+5} q_i^a = 0$ for the ADE elliptic $K3$ requires that

$$q_{r+6}^a + q_{r+7}^a + q_{r+8}^a = 0, \quad \forall a. \tag{5.7}$$

The vacuum energy of this $N=2$ sigma model is given by the D-flatness equations

$$\sum_{j=1}^{r+8} Q_j^a |\phi_j|^2 = \rho^a, \quad a=0,\dots,r, \tag{5.8}$$

where this space of solutions, up to the identifications imposed by the action of gauge group, has a toric realization. This represented by $(r+8)$ vertices V_j $(j = 1, \ldots, r+8)$ of the standard lattice $\mathbf{Z^5}$ satisfying the following toric relations:

$$\sum_{j=1}^{r+8} Q_j^a V_j = 0, \quad a = 0, \ldots, r. \tag{5.9}$$

Using the conventional notation $V_j = V_{j\ell}$, $\ell = 1, \ldots, 5$, the above toric data (5.9) may be split as

$$\begin{aligned} \sum_{i=1}^{r+5} q_i^a V_{i\ell'} &= 0, \quad \ell' = 1,2,3 && (5.10) \\ q_{r+6}^a V_{r+6\ell'} + q_{r+7}^a V_{r+7\ell'} + q_{r+8}^a V_{r+8\ell'} &= 0, \quad \ell' = 1,2,3 && (5.11) \\ \sum_{j=1}^{r+8} Q_j^a V_{j4} &= 0, && (5.12) \\ \sum_{j=1}^{r+8} Q_j^a V_{j5} &= 0. && (5.13) \end{aligned}$$

Equation (5.10) is nothing but the equation (3.18) where

$$V_{i\ell'} = v_i = (n_i, m_i, s_i), \quad \ell' = 1,2,3, \tag{5.14}$$

To write down the LG mirror superpotential, we follow the same analysis used in section 3. This is obtained in terms of new gauge invariant monomials:

$$x^j = \prod_{\ell=1}^{5} x_\ell^{V_{j\ell}}. \tag{5.15}$$

Thus the ADE mirror superpotentials are

$$\sum_{j=1}^{r+8} a_j \prod_{\ell=1}^{5} x_\ell^{V_{j\ell}} = 0. \tag{5.16}$$

However to work out the explicit form of this equation, we have to solve the toric constraint equations (5.10-13). A solution of these toric data is given

$$\begin{aligned} V_i &= (n_i, m_i, s_i, 0, 0), \quad i = 1, \ldots, r+5 && (5.17) \\ V_{r+6} &= (0,0,0,\alpha,\alpha') && (5.18) \\ V_{r+7} &= (0,0,0,\beta,\beta'), && (5.19) \\ V_{r+8} &= (0,0,0,\gamma,\gamma') && (5.20) \end{aligned}$$

where $\alpha, \beta, \gamma, \alpha', \beta'$ and γ' are six integers satisfying

$$\begin{aligned} q^a_{r+6}\alpha + q^a_{r+7}\beta + q^a_{r+8}\gamma &= 0, \quad (5.21) \\ q^a_{r+6}\alpha' + q^a_{r+7}\beta' + q^a_{r+8}\gamma' &= 0. \quad (5.22) \end{aligned}$$

For latter use, we choose a special case where

$$(q^a_{r+6}, q^a_{r+7}, q^a_{r+8}) = (1, -2, 1), \quad \forall a. \quad (5.23)$$

In this way, a naive solution of equation (5.21-22) is given by

$$(\alpha, \beta, \gamma) = (1, 1, 1), \quad (5.24)$$

and

$$(\alpha', \beta', \gamma') = (-1, 0, 1). \quad (5.25)$$

Taking this special case, we get the following LG mirror superpotential

$$\sum_{i=1}^{r+5} a_i {x_1}^{n_i} {x_2}^{m_i} {x_3}^{s_i} + x_4 (\frac{a_{r+6}}{x_5} + a_{r+7} + a_{r+8} x_5) = 0. \quad (5.26)$$

However using equations (5.8) and (5.23), the mirror map for $q^a_i = 0$ ($i = 1, \ldots r+5$), breaking the $U(1)^{r+1}$ symmetry to $U(1)$, implies that the LG fields corresponding to the mirror base geometry are constrained by

$$\frac{a_{r+6}}{x_5} + a_{r+7} + a_{r+8} x_5 = 0. \quad (5.27)$$

This means that the mirror base geometry is zero-dimensional space. From this requirement, the mirror geometry of ADE hypersurfaces, obtained after introducing the non dynamical fields, are given by

$$\sum_{i=1}^{r+5} a_i {x_1}^{n_i} {x_2}^{m_i} {x_3}^{s_i} = uv \quad (5.28)$$

where $\sum_{i=1}^{r+5} a_i {x_1}^{n_i} {x_2}^{m_i} {x_3}^{s_i} = 0$ is the equation of the ADE elliptic K3 surfaces. Finally if we consider F-theory compactifications on these elliptic hypersurfaces, we obtain $D = 4$ $N = 1$ ADE gauge theories with non matter.

6 Conclusion

In this paper, we have studied the Landau Ginzburg theory mirror to $2D$ $N = 2$ gauged linear toric sigma model. We have derived new classes for elliptic Calabi-Yau superpotentials of Landau Ginzburg theories. The latters play a crucial role in string/ brane physics. In the Calabi-Yau threefolds case, we have considered two examples of the mirror symmetry for toric sigma model. First, we have given the mirror theory of linear sigma model on the canonical line bundle over the Hirzebruch surfaces $\mathbf{F_n}$, recovering the leading example of $\mathbf{F_0}$ studied in the context of the mirror action of Lagrangian D-branes [11]. In this case, we have shown that the mirror geometry is an elliptic Calabi-Yau threefolds whose the fiber is a Reimann surface. Second we have found a special elliptic and $K3$ fibration Calabi-Yau threefolds extending the elliptic $K3$, considered in the geometric engineering of $4D$ $N = 2$ superconformal field theory, to an elliptic Calabi-Yau threefolds with affine so(8) Lie algebra Mori vectors. Moreover, this geometry gives a new $N = 1$ SO(8) pure Yang-Mills theory in six dimensions from the F-theory compactification which may be associated to non perturbative gauge symmetry in the heterotic string picture [42]. Finally, we have used the interplay between toric geometry and gauged linear sigma model to derive an intuitive algebraic realization for the mirror superpotentials associated to ADE Calabi-Yau fourfolds hypersurfaces.

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